Deriving the equations of motion

Before we start to derive the equations of motion, it is important to make sure we know the standard symbols for describing the movement of an object. Conventionally, we use the following symbols to represent the properties of an object.

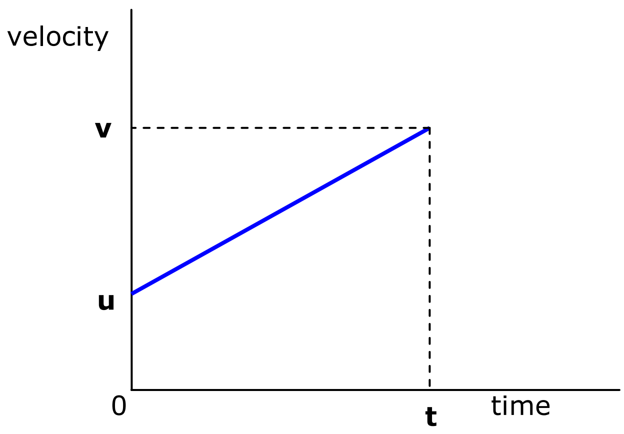
property	symbol
displacement	s
initial velocity	u
final velocity	v
time	t
acceleration	a

Now let's imagine an object travelling with velocity **u** ms^{-1}.

If our object experiences an unbalanced force, this will cause the object to accelerate at **a** ms^{-2}. The force acts for a time **t** and so the acceleration must also last for time **t**.

After **t** seconds, the object has final velocity **v** ms^{-1}.

We can show this motion on a velocity-time graph.



The gradient of the velocity-time graph gives us the acceleration, **a**.

$$\text{gradient} = \frac{\text{change in velocity}}{\text{time}} = \frac{v - u}{t} = a$$

This equation can be rearranged to obtain our first equation of motion.

Deriving the equations of motion

Starting with

$$a = \frac{v - u}{t}$$

We can multiply both sides by **t** to remove the fraction

$$v - u = at$$

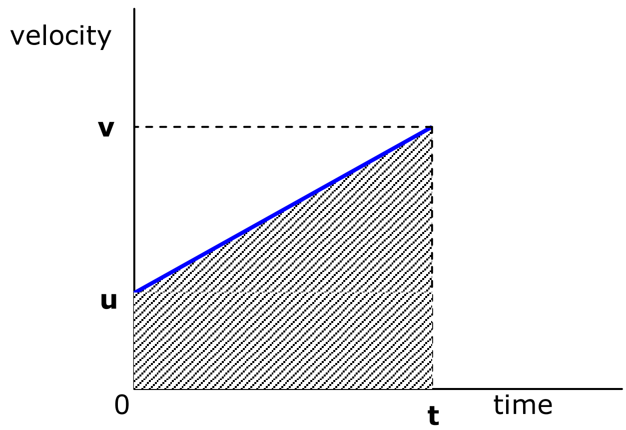
and moving the (-u) to the other side gives

$$\boxed{v = u + at}$$

This is our first equation of motion.

Let's go back to the velocity-time graph we created earlier and see if we can obtain an equation for the displacement **s** of the object.

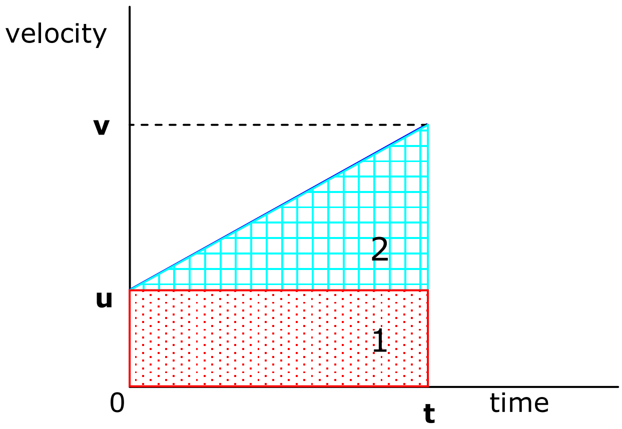
The area under a velocity-time graph is equal to the displacement of the object. So if we can calculate this shaded area, we will find the displacement.



The simplest way to do calculate the shaded area is to split it up into smaller shapes. Try to choose shapes with simple area equations, such as rectangles and triangles, as shown on the next diagram.

Deriving the equations of motion

Here we have split the shaded area into a triangle and a rectangle.



The sum of areas 1 and 2 is equal to the displacement.

Let's calculate the areas individually.

<u>Area 1</u>

Area = ut

<u>Area 2</u>

Area = ½ x base x height

= ½ x t x (v-u)

but we know from earlier that v-u = at

so

= ½ x t x at

Area = $\frac{1}{2}at^2$

The total displacement **s** is the sum of areas 1 & 2.

s = area 1 + area 2

and so

$$s = ut + \frac{1}{2}at^2$$

This is our second equation of motion.

Deriving the equations of motion

We will use both of the equations we have obtained so far to reach the third equation of motion. This will require a bit of algebra so I will only make small changes in each line so you can see what is happening

Notice that both

s = ut + ½at² and **v = u + at** include the time variable **t**.

There will be some situations when we do not have any information about time and so it would be a good idea to derive an equation that does not have a **t** term.

To do this, we rearrange our first equation to get

$$t = \frac{v-u}{a}$$

and use this to replace **t** wherever it appears in the second equation. So

$$s = ut + \tfrac{1}{2}at^2$$

becomes

$$s = u\left(\frac{v-u}{a}\right) + \frac{1}{2}a\left(\frac{v-u}{a}\right)^2$$

start by taking **a** out of squared bracket

$$s = u\left(\frac{v-u}{a}\right) + \frac{1}{2}a\frac{(v-u)^2}{a^2}$$

now cancel out the **a** terms

$$s = u\left(\frac{v-u}{a}\right) + \frac{1}{2}\frac{(v-u)^2}{a}$$

remove fractions by multiplying **both sides** of equation by **2a**

$$2as = 2u(v-u) + (v-u)^2$$

multiply out the brackets

$$2as = 2uv - 2u^2 + v^2 - 2uv + u^2$$

collect & simplify common terms

$$2as = v^2 - u^2$$

So

$$v^2 = u^2 + 2as$$

is our third equation of motion.